MECHANICS OF STRUCTURAL INSTABILITY IN THIN-WALLED STRUCTURES

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ABSTRACT

Instability is an important branch of structural mechanics which examines alternate equilibrium states associated with large deformation. Structural instability in one dimensional beam-column members and two dimensional plate members is examined in this paper. Various forms of buckling, namely, column buckling, beam buckling and local plate buckling, under different loading and boundary conditions are illustrated to establish the phenomena as well as various analysis and design methods for practical design.

KEYWORDS

Structural instability, column buckling, beam buckling, local plate buckling.

COLUMN BUCKLING

Consider a column pinned at both ends [1] as shown in Figure 1.

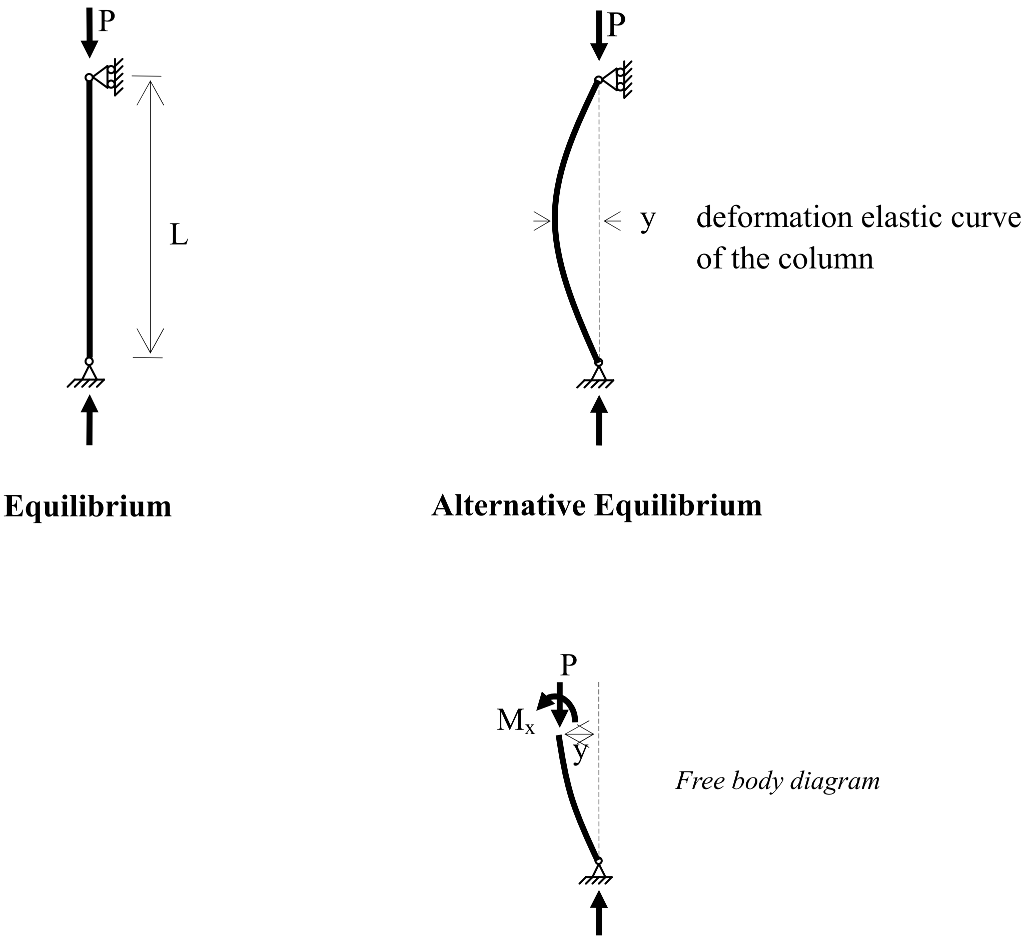


Figure 1 Buckling of a column

Taking moment about the pinned support in the free body diagram,

$$P \cdot y + M_x = 0$$

As $M_x = EI\dfrac{d^2y}{dx^2}$,

$$EI\frac{d^2y}{dx^2} + P \bullet y = 0$$

By writing $k^2 = \dfrac{P}{E\,I}$, the above equation is arranged as follows:

$$\frac{d^2 y}{dx^2} + k^2 \bullet y = 0$$

The solution of the differential equation is given by:

$$y = A \sin(kx) + B \cos(kx) + C\,x + D$$

Applying the boundary conditions at both pinned ends:

$$y = \frac{d^2 y}{dx^2} = 0 \qquad \text{when } x = 0 \text{ and } x = L$$

$\Rightarrow \quad B = C = D = 0, \qquad$ and $\qquad \sin(kL) = 0$

Hence,

$$kL = n\pi \qquad \text{and} \qquad \frac{n\,\pi}{L} = \sqrt{\frac{P}{E\,I}}\ .$$

By taking n = 1 as a minimum, the elastic critical buckling load, P_E , is given by:

$$P_E = \frac{\pi^2\,E\,I}{L^2}$$

The deflected shape for the elastic critical buckling of the column is given by:

$$y = A \sin\left(\frac{\pi\,x}{L}\right)$$

where A is a constant with an arbitrary value.

Design method using the slenderness of a column

Expanding the second moment of area, I , as the product of area and the radius of gyration, $A\,r_y^{\,2}$, the elastic critical buckling load, P_E , may be re-presented as follows:

$$P_E = \frac{\pi^2}{L^2} E\,A\,r_y^{\,2}$$

Then the elastic critical buckling strength, p_E , is given as follows:

$$p_E = \frac{P_E}{A} = \frac{\pi^2}{\left(\frac{L}{r_y}\right)^2} E$$

Re-writing the column slenderness $\lambda = \frac{L}{r_y}$ as $\lambda = \frac{L}{r_y}$, the elastic critical buckling strength, p_E, is given as follows:

$$p_E = \frac{\pi^2 E}{\lambda^2}$$

This equation gives the elastic critical buckling strength of a column in terms of its material property, E, as well as its geometrical property, i.e. the column slenderness.

It is important to evaluate the compressive buckling strength of real columns, p_c, in the presence of initial mechanical and geometrical imperfections, a Perry Robertson interaction formula [2] is adopted as follows:

$$p_c = \frac{p_E\, p_y}{\phi + \sqrt{\phi^2 - p_E\, p_y}} \qquad \text{where} \quad \phi = \frac{p_y + (1+\eta)p_E}{2}$$

where

p_y is the design strength of the column member;

η is the Perry factor;

$= 0.001\, a\ (\lambda - \lambda_o)$

a is the Robertson constant with typical values such as 2.0, 3.5, 5.5, 8.0;

λ_o is the limiting slenderness;

$= 0.2\, \pi \sqrt{\frac{E}{p_y}}$

The choice of the value of the Robertson constant a depends on the cross-sections of the columns, the axes of buckling, and the thicknesses of the sections.

Design method using non-dimensionalised slenderness of a column

It is highly desirable to represent the elastic critical buckling strength of a column, p_E , into a non-dimensional ratio. Consider the elastic critical buckling strength, p_E , as follows:

$$p_E = \frac{\pi^2 E}{\lambda^2}$$

Dividing both sides of the equation by p_y ,

$$\frac{p_E}{p_y} = \frac{\pi^2 E}{p_y} \frac{1}{\lambda^2}$$

It is useful to establish a material parameter λ_Y as follows:

$$\lambda_Y = \pi \sqrt{\frac{E}{p_y}} \qquad \text{so that} \qquad \lambda_Y^{\,2} = \frac{\pi^2 E}{p_y}$$

Hence

$$\frac{p_E}{p_y} = \frac{\lambda_Y^{\,2}}{\lambda^2}$$

Similarly, a strength reduction factor may be established to allow for column buckling in real columns, and a number of non-dimensionalised column buckling curves are established to provide strength reduction factors to the design strengths of real columns for practical design as shown in Figure 2. The following parameters are adopted:

$\bar{\chi}$ is the strength reduction factor due to column buckling;

$$= \frac{p_c}{p_y}$$

$\bar{\lambda}$ is the non-dimensionalised slenderness of a column;

$$= \frac{\lambda}{\lambda_Y}$$

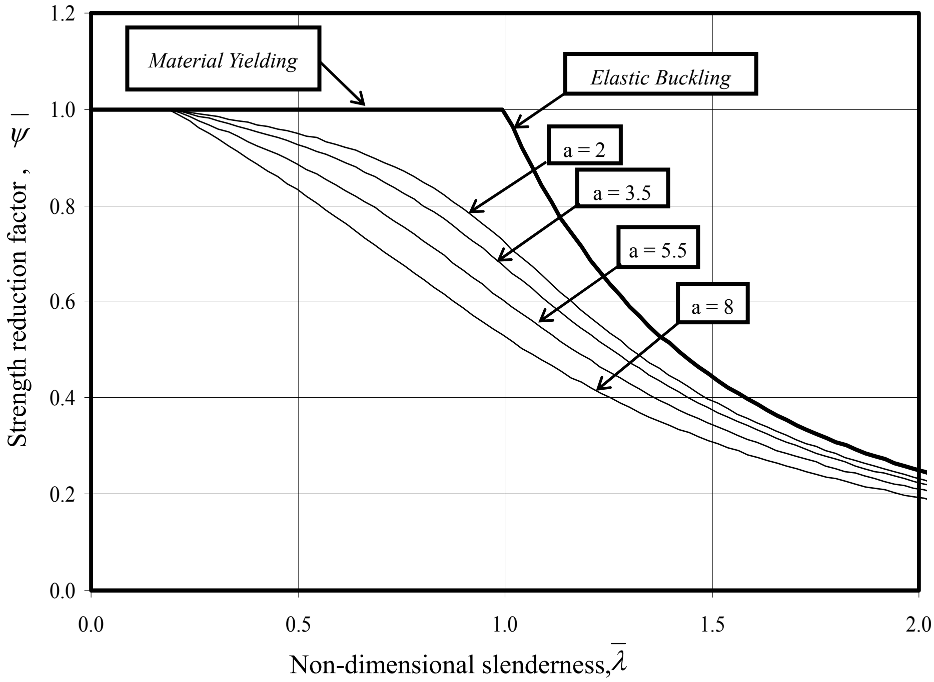


Figure 2 Non-dimensionalized column buckling curves

BEAM BUCKLING

For a beam member with a bi-symmetric cross-section under uniform moment, the elastic critical buckling moment, M_E, is given by [2]:

$$M_E = \frac{\pi}{L}\sqrt{\frac{EI_yGJ}{\gamma}}\sqrt{\left(1+\frac{\pi^2 EH}{L^2 GJ}\right)}$$

where

- L is the beam length;
- E is the Young's modulus;
- G is the shear modulus;
- I_y is the second moment of area about the minor axis;
- H is the warping constant;
- J is the torsional constant; and

$$\Gamma = 1 - \frac{I_y}{I_x}$$

Design method using the slenderness of a beam member

Similar to a column member, it is possible to evaluate the buckling moment, M_b , of a beam with the Perry-Robertson interaction formula and the equivalent slenderness, λ_{LT} , of the beam [2] as follows:

$M_b = p_b S_x$

where

$$p_b = \frac{p_E \, p_y}{\phi_{LT} + \sqrt{\phi_{LT}^{2} - p_E \, p_y}}$$

where

$$\varphi_{LT} = \frac{p_y + (1 + \eta_{LT}) p_E}{2}$$

$$\eta_{LT} = 0.001 \, \alpha_{LT} \, (\lambda_{LT} - \lambda_{L0})$$

α_{LT} is the Robertson constant which is taken as 7.0;

λ_{L0} is the non-dimensionalised limiting slenderness;

$$= 0.4\pi \sqrt{\frac{E}{p_y}}$$

$$p_E = \frac{\pi^2 E}{\lambda_{LT}^{2}}$$

$$\lambda_{LT} = u \, v \, \lambda$$

$$\lambda = \frac{L}{r_y}$$

$$u = \left(\frac{I_y S_x^{2} \gamma}{A^2 H} \right)^{0.25}$$

$$x = \pi \sqrt{\frac{1 + \upsilon}{10}} \sqrt{\frac{A \, H}{J \, I_y}}$$

$$v = \frac{1}{\sqrt[4]{1 + \frac{1}{20} \left(\frac{\lambda}{x} \right)^2}}$$

S_x is the plastic modulus about the major axis;

υ is the Poisson's ratio.

Design method using non-dimensionalised slenderness of a beam

By adopting the non-dimensionalised slenderness of a beam, a non-dimensionalised beam buckling curve may be established as shown in Figure 3 to provide strength reduction factors to the design strengths or the moment capacities of real beams for practical design. The following parameters are adopted:

$\overline{\chi}_{LT}$ is the strength reduction factor due to beam buckling;

$$= \frac{p_b}{p_y}$$

$\overline{\lambda}_{LT}$ is the non-dimensionalised slenderness of a beam;

$$= \frac{\lambda_{LT}}{\lambda_Y}$$

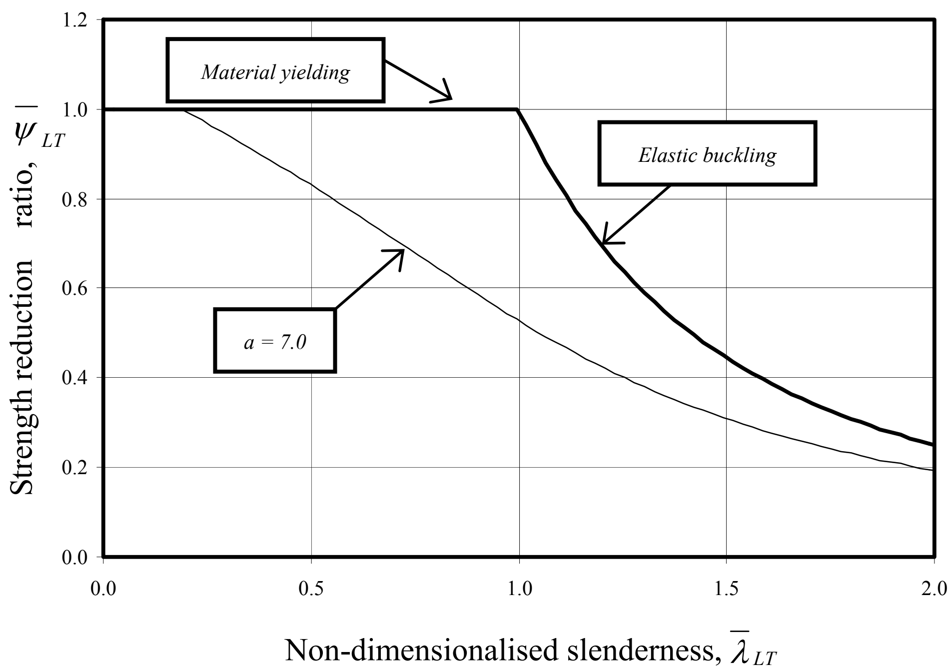


Figure 3 Non-dimensionalised beam buckling curve

LOCAL PLATE BUCKLING UNDER UNIFORM COMPRESSION

Consider a rectangular plate which is simply supported along both the longitudinal and the transverse edges [3]. The plate is under a compression force of N_o along the transverse edges.

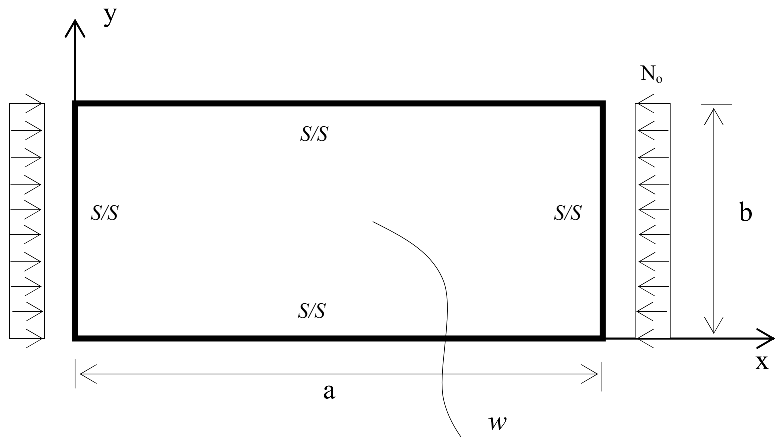


Figure 4 Local buckling in a simply supported plate under compression

Assume that the deflected shape of the plate is represented with sine curves in both longitudinal and traverse directions as follows:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(m\,\pi\,\frac{x}{a}\right)\sin\left(n\,\pi\,\frac{y}{b}\right)$$

where A_{mn} is constant to be determined.

The strain energy is given by:

$$U = \pi^4\, a\,\frac{b}{8}\,D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}{}^2$$

The work done by the external force is given by:

$$\frac{1}{2}N_x\int_0^a\int_0^b\left(\frac{dw}{dx}\right)^2 dx\,dy = \pi^2\,\frac{b}{8a}N_o\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2\,A_{mn}{}^2$$

By equating the work done and the strain energy as follows:

$$\pi^2\,\frac{b}{8a}N_{cr}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2\,A_{mn}{}^2 = \pi^4\,a\,\frac{b}{8}\,D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}{}^2$$

The elastic critical buckling load, N_{cr} , is found to be as follows:

$$N_{cr} = \frac{\pi^2 a^2 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 A_{mn}{}^2}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2 A_{mn}{}^2}$$

To obtain the smallest value of N_{cr}, consider only the first term as follows:

$$N_{cr} = \left(\frac{\pi^2 a^2 D}{m^2}\right)\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

For the minimum value of N_{cr}, n = 1

$$N_{cr} = \left(\pi^2 \frac{D}{a^2}\right)\left(m+\frac{a^2}{m^2 b^2}\right)^2$$

Alternatively, by re-writing the formula in term of the elastic critical buckling strength, p_{cr} ,

$$p_{cr} = \frac{N_{cr}}{t} \quad \text{or} \quad = K\frac{\pi^2 D}{b^2 t}$$

$$= \frac{\pi^2 E}{12(1-\upsilon^2)} K\left(\frac{t}{b}\right)^2$$

where k is equal to 4 for the minimum value of the elastic critical buckling strength in simply supported plates.

LOCAL PLATE BUCKLING UNDER IN-PLANE BENDING

Consider a rectangular plate which is simply supported along both the longitudinal and the transverse edges [3]. The plate is under an in-plane bending moment with a linearly varying compression force of N_o along the transverse edges.

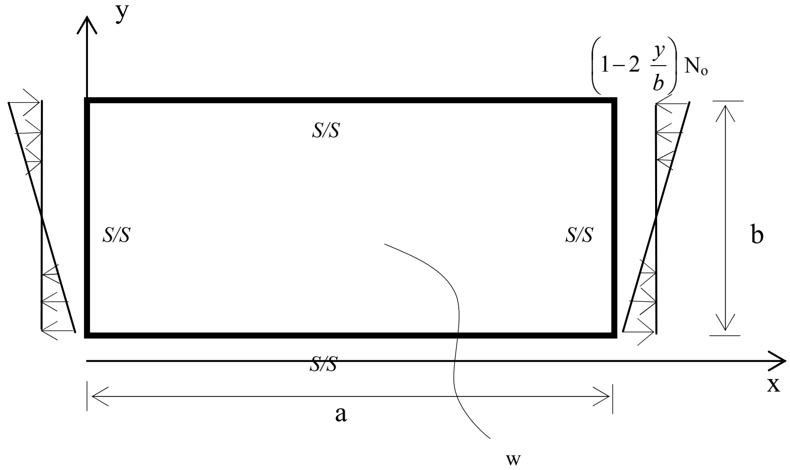


Figure 5 Local buckling in a simply supported plate under in-plane bending

Assume that the deflected shape of a plate is represented with sine curves in both longitudinal and traverse directions as follows:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where A_{mn} is constant to be determined.

The strain energy is given by:

$$U = \frac{D}{2}\frac{ab\pi^4}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^{2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2$$

The work done by the external force is given by:

$$T = \frac{1}{2}\int_0^a\int_0^b N_0\left(1-\alpha\frac{y}{b}\right)\left(\frac{dw}{dx}\right)^2 dxdy \qquad \text{where } \alpha = 2 \text{ for pure bending}$$

By equating the work done and the strain energy, the elastic critical buckling load is given by:

$$N_{cr} = \frac{\pi^4 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^{\ 2} (\frac{m^2}{a^2} + \frac{n^2}{b^2})^2}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^{\ 2} \frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}[\sum_{n=1}^{\infty} A_{mn}^{\ 2} - \frac{32}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}^{\infty}\frac{A_{mn}A_{mi}ni}{(n^2-i^2)^2}]}$$

The derivative with respect to A_{mn} is given by:

$$DA_{mn}\pi^4 (\frac{m^2}{a^2} + \frac{n^2}{b^2})^2 = (N_0)_{cr} \{A_{mn} \frac{m^2\pi^2}{a^2} - \frac{\alpha}{2}\frac{m^2\pi^{\ 2}}{a^2}[A_{mn} - \frac{16}{\pi^2}\sum_{i}^{\infty}\frac{A_{mi}ni}{(n^2-i^2)^2}]\}$$

Assume that the deflected shape is approximated by the first three terms:

$$w = \sin\frac{m\pi x}{a}\sum_{n=1}^{3} A_{mn}\sin\frac{n\pi y}{b}$$

The minimum value of the elastic critical buckling strength, p_{cr} , is given by:

$$p_{cr} = \frac{N_{cr}}{t} \quad \text{or} \quad = K\frac{\pi^2 D}{b^2 t}$$

$$= \frac{\pi^2 E}{12(1-\upsilon^2)} K \left(\frac{t}{b}\right)^2$$

The value of K is found to be dependent on the value of $\frac{a}{b}$ as follows:

$\frac{a}{b}$	0.4	0.5	0.6	0.667	0.75	0.8	0.9	1.0	1.5
k	29.1	25.6	24.1	23.9	24.1	24.4	25.6	25.6	24.1

DESIGN METHODS USING EFFECTIVE SECTIONS

It is important to evaluate the effective section capacities of plates undergoing local plate buckling in the presence of initial mechanical and geometrical imperfections, and a codified section analysis using an effective thickness approach or an effective width approach may be adopted.

Based on the effective width approach, the effective width, b_{eff} , of a plate undergoing local plate buckling may be established [4] as follows:

$$\frac{b_{eff}}{b} = \frac{1}{\sqrt[5]{\left[1+14\left[\sqrt{f_c/p_{cr}}-0.35\right]^4\right]}}$$

where b is the width of the plate and f_c is the applied compressive stress of the plate.

Figure 6 plots the function of $\frac{b_{eff}}{b}$ against the value of $\frac{b}{t}$.

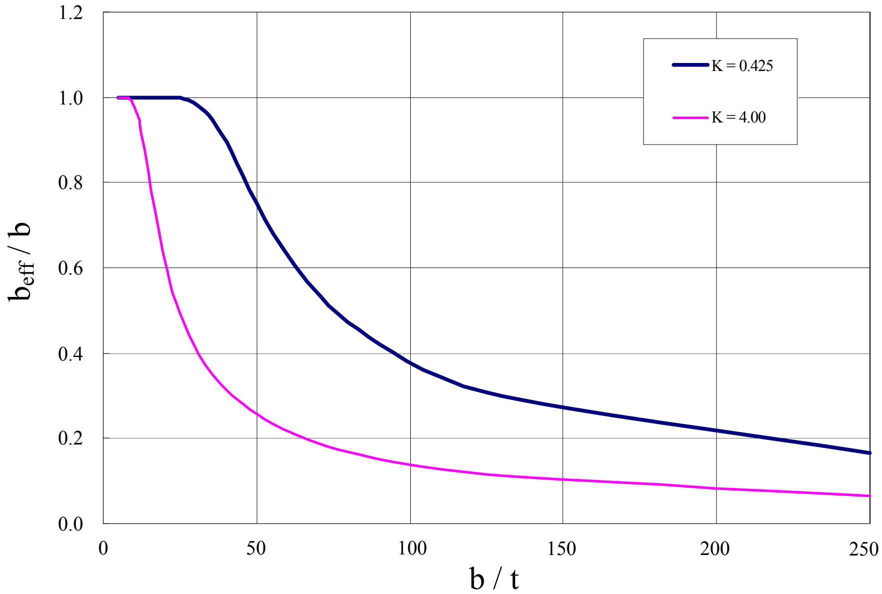


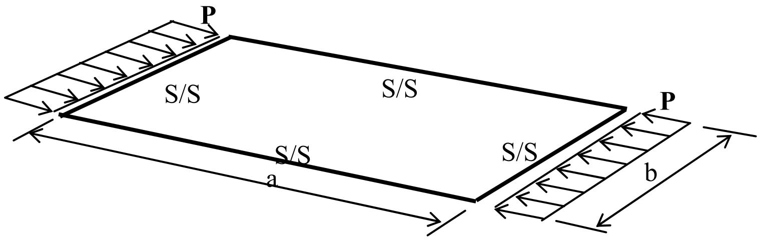
Figure 6 Ratio of effective width to flat width of compression plate

COMPARSION WITH NUMERICAL MODELS

Figures 7, 8 and 9 illustrate local plate buckling in rectangular plates under different loading and boundary conditions. The following analyses are performed and summarised for direct comparison:

- analytical analysis using energy method;
- section analysis using effective width approach;
- elastic critical buckling analysis using finite element technique [5]; and
- geometrical and material nonlinear analysis using finite element technique [5].

Geometry



Geometrical data

a = 200 mm

b = 100 mm

t = 1.6 mm

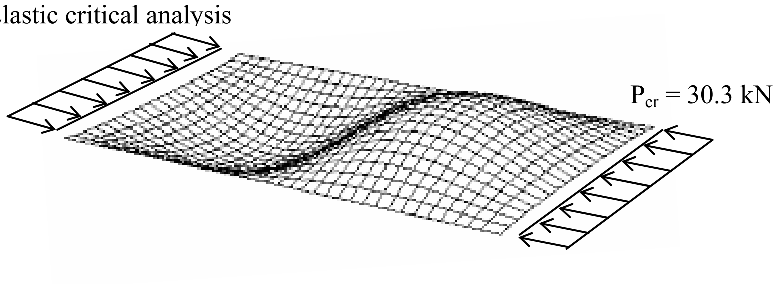
φ = a/b

= 2

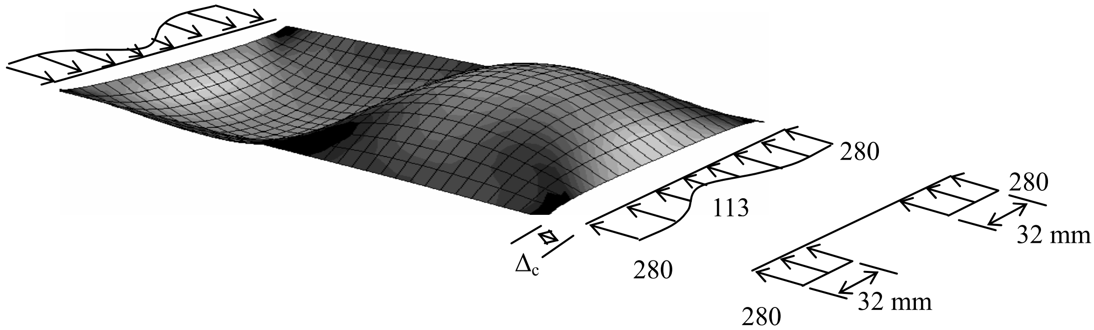
b/t = 62.5

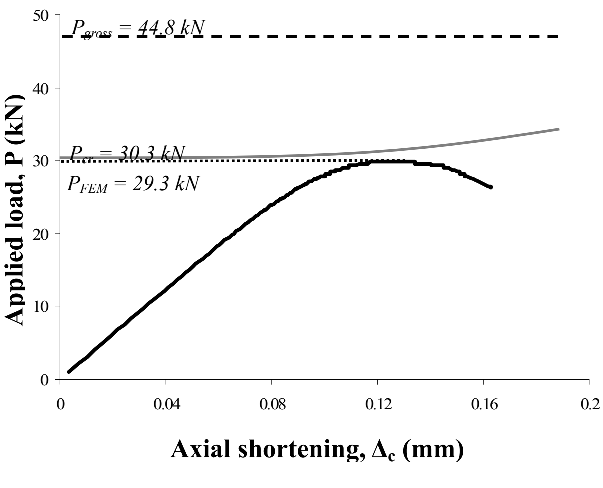
Material data

Elastic critical analysis



Material and geometrical nonlinear analysis





$$p_{cr} = \frac{\pi^2 E}{12(1-\upsilon^2)} K \left(\frac{t}{b}\right)^2$$

$K = 4.0$

$p_{cr} = 189.4\ \text{N/mm}^2$

$$\frac{b_{eff}}{b} = \left[1 + 14\left(\sqrt{\frac{f_c}{p_{cr}}} - 0.35\right)^4\right]^{-0.2}$$

$= 0.64$

$P_{gross} = A \times p_y$

$= 44.8$ kN

$P_{cr} = 30.3$ kN

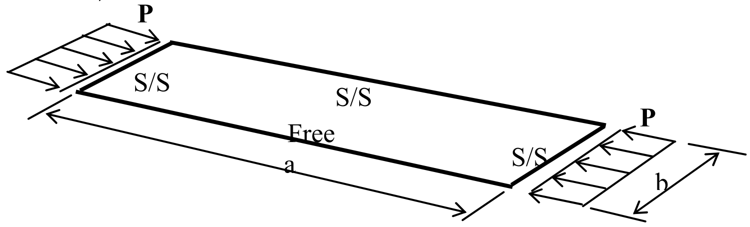
$P_{LB} = A_{eff} \times p_y$

$= 28.9$ kN

$P_{FEM} = 29.3$ kN

Figure 7 Local plate buckling in a plate under compression. Simply supported along four sides.

Geometry



Geometrical data

a = 200 mm

b = 50 mm

t = 1.6 mm

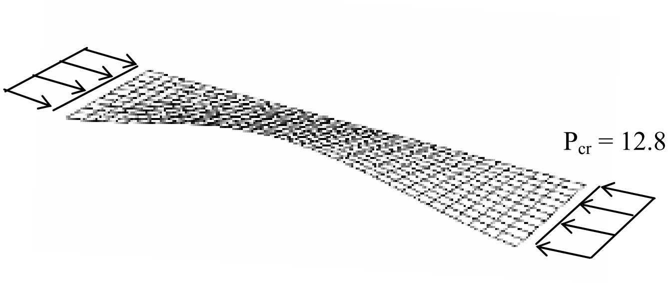
φ = a/b

= 1

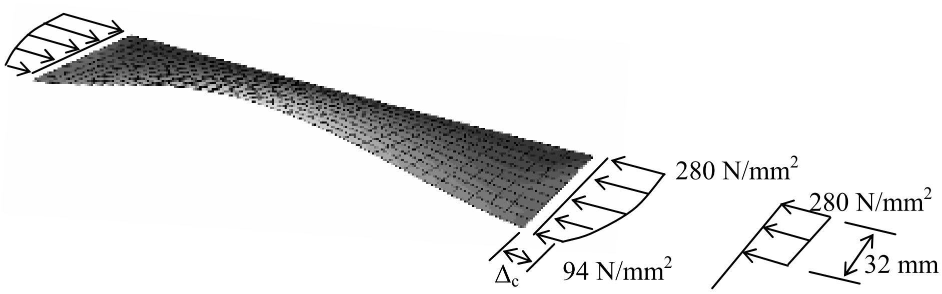
b/t = 31.3

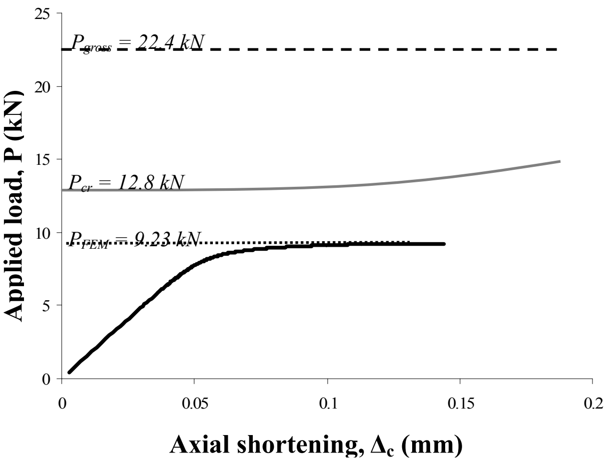
Material data

Elastic critical analysis



Material and geometrical nonlinear analysis





$$p_{cr} = \frac{\pi^2 E}{12(1-\upsilon^2)} K \left(\frac{t}{b}\right)^2$$

$K = 0.423$

$p_{cr} = 189.4\ \text{N/mm}^2$

$$\frac{b_{eff}}{b} = \left[1 + 14\left(\sqrt{\frac{f_c}{p_{cr}}} - 0.35\right)^4\right]^{-0.2}$$

$= 0.647$

$P_{gross} = A \times p_y$

$= 22.4$ kN

$P_{cr} = 12.8$ kN

$P_{LB} = A_{eff} \times p_y$

$= 9.45$ kN

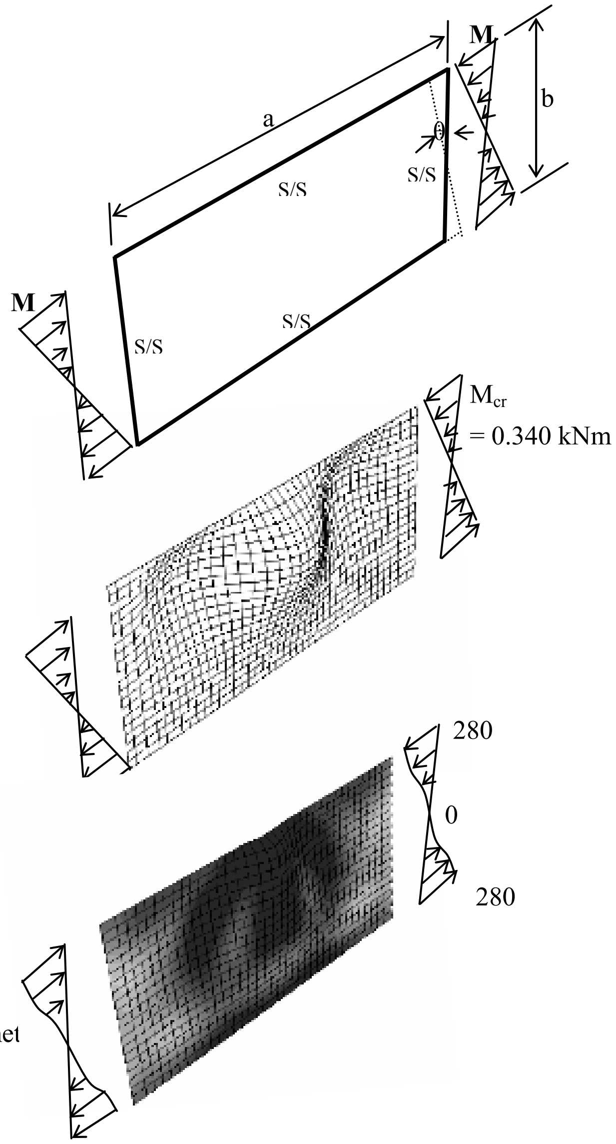
$P_{FEM} = 9.23$ kN

Figure 8 Local plate buckling in a plate under compression.
Simply supported along three sides and free along one side.

Geometry

Elastic critical analysis

Material and geometric nonlinear analysis



Geometrical data

a = 200 mm

b = 100 mm

t = 0.6 mm

φ = a/b

= 2

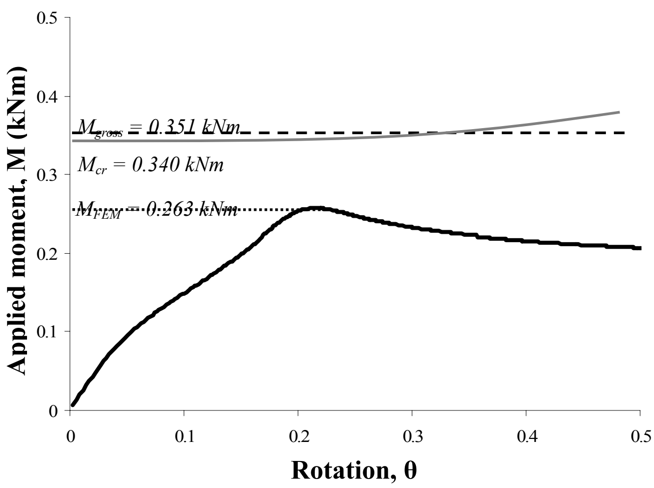
b/t = 62.5

Material data

p_y = 350 N/mm^2

E = 205 kN/mm^2

ν = 0.3



$$p_o = \left[1.13 - 0.0019\frac{b}{t}\left(\frac{p_y}{280}\right)^{0.5}\right]p_y$$

$$= 271.6 \text{ N/mm}^2$$

M_{gross} = $p_y \times Z$

= 0.351 kNm

M_{cr} = 0.340 kNm

M_{LB} = $p_o \times Z$

= 0.272 kNm

M_{FEM} = 0.263 kNm

Figure 9 Local plate buckling in a plate under in-plane bending.
Simply supported along four sides.

Comparison with gross section capacities and effective section capacities in the presence of local plate buckling are provided. It should be noted that the results from various analyses are very close among themselves.

CONCLUSIONS

It is important to appreciate structural instability in one dimensional beam-column members and two-dimensional plate members under different loading and boundary conditions. Some basic features of column buckling, beam buckling and local plate buckling are presented to facilitate buckling analysis and design in practice.

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